Slack variety of a polytope and its applications

João Gouveia

19th of October 2018 - ICERM Workshop on Real Algebraic Geometry and Optimization
Co-authors: past, present and future

Kanstantsin Pashkovich - University of Waterloo
Richard Z. Robinson - Microsoft
Rekha Thomas - University of Washington
Antonio Macchia - Universitá degli Studi di Bari
Amy Wiebe - University of Washington
Jeffrey Pang - National University of Singapore
Ting Kei Pong - Hong Kong Polytechnic University
Section 1

Polytopes and their realization spaces
A polytope is:
a convex hull of a finite set of points in $\mathbb{R}^n$.

\[ P = \text{conv}\{p_1, p_2, \ldots, p_v\} \]

$\mathcal{V}$-representation
A polytope is:
a convex hull of a finite set of points in $\mathbb{R}^n$.

$$P = \text{conv}\{p_1, p_2, \ldots, p_v\}$$

$\mathcal{V}$-representation

a compact intersection of half spaces in $\mathbb{R}^n$.

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

$\mathcal{H}$-representation
A polytope is: a convex hull of a finite set of points in $\mathbb{R}^n$.

$$P = \text{conv}\{p_1, p_2, \ldots, p_v\}$$

$\mathcal{V}$-representation

A face of $P$ is its intersection with a supporting hyperplane, and the set of faces ordered by inclusion forms the face lattice of $P$.

A compact intersection of half spaces in $\mathbb{R}^n$.

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

$\mathcal{H}$-representation
We say that two polytopes are **combinatorially equivalent** if they have the same face lattice.
We say that two polytopes are \textit{combinatorially equivalent} if they have the same face lattice.

Given a combinatorial class of polytopes, we call each polytope in that class a \textit{realization} of that class.

We will call the space of all realizations of the combinatorial class of a polytope \( P \) the \textit{realization space} of \( P \).
We say that two polytopes are **combinatorially equivalent** if they have the same face lattice.

Given a combinatorial class of polytopes, we call each polytope in that class a **realization** of that class.

We will call the the space of all realizations of the combinatorial class of a polytope $P$ the **realization space** of $P$.

**Question:** How do we make such an object concrete?
The classic model for the realization space

There is a very direct way of modelling the realizations space.

Given a $d$-polytope $P$, define $R(P)$ to be the set of all $Q \in \mathbb{R}^{d \times v}$ such that the convex hull of their columns is combinatorially equivalent to $P$. 

$$R(P) = \{ [w_1 x_1 y_1 z_1, w_2 x_2 y_2 z_2] : w, x, y, z \text{ are vertices of a square} \}$$

We can also mod out affine transformations by fixing an affine basis $B$. 

$$R(P, B) = \{ [0 0 1 x_1, 1 0 0 x_2] : e_1, 0, e_2, x \text{ are vertices of a square} \} = \{ x \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \geq 1 \}$$
The classic model for the realization space

There is a very direct way of modelling the realizations space.

Given a \(d\)-polytope \(P\) define \(\mathcal{R}(P)\) to be the set of all \(Q \in \mathbb{R}^{d \times v}\) such that the convex hull of their columns is combinatorially equivalent to \(P\).
There is a very direct way of modelling the realizations space.

Given a $d$-polytope $P$ define $\mathcal{R}(P)$ to be the set of all $Q \in \mathbb{R}^{d \times v}$ such that the convex hull of their columns is combinatorially equivalent to $P$.

\[
\mathcal{R}(P) = \{ [w_1 x_1 y_1 z_1; w_2 x_2 y_2 z_2] : w, x, y, z \text{ are vertices of a square} \}
\]

We can also mod out affine transformations by fixing an affine basis $B$.

\[
\mathcal{R}(P, B) = \{ [0 0 1 x_1; 1 0 0 x_2] : e_1, 0, e_2, x \text{ are vertices of a square} \} = \{ x \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \geq 1 \}
\]
The classic model for the realization space

There is a very direct way of modeling the realizations space. Given a $d$-polytope $P$ define $\mathcal{R}(P)$ to be the set of all $Q \in \mathbb{R}^{d \times v}$ such that the convex hull of their columns is combinatorially equivalent to $P$.

$$\mathcal{R}(P) = \left\{ \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{bmatrix} : w, x, y, z \text{ are vertices of a square} \right\}$$
The classic model for the realization space

There is a very direct way of modelling the realizations space. Given a $d$-polytope $P$ define $\mathcal{R}(P)$ to be the set of all $Q \in \mathbb{R}^{d \times v}$ such that the convex hull of their columns is combinatorially equivalent to $P$.

$$\mathcal{R}(P) = \left\{ \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{bmatrix} : w, x, y, z \text{ are vertices of a square} \right\}$$

We can also mod out affine transformations by fixing an affine basis $B$.

$$\mathcal{R}(P, B) = \left\{ \begin{bmatrix} 0 & 0 & 1 & x_1 \\ 1 & 0 & 0 & x_2 \end{bmatrix} : e_1, 0, e_2, x \text{ are vertices of a square} \right\}$$

$$= \{ x \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \geq 1 \}$$
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;

...
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;
- They are universal even for 4-polytopes [Richter-Gebert 96];
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;
- They are universal even for 4-polytopes [Richter-Gebert 96];
- The modding out of transformations is very basis dependent;
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;
- They are universal even for 4-polytopes [Richter-Gebert 96];
- The modding out of transformations is very basis dependent;
- It is not invariant under duality;
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;
- They are universal even for 4-polytopes [Richter-Gebert 96];
- The modding out of transformations is very basis dependent;
- It is not invariant under duality;
- They are difficult to compute with.
Properties of the classic model

These realization spaces are well-studied, and much is known about them.

- They are very natural;
- They are semialgebraic;
- They are universal even for 4-polytopes [Richter-Gebert 96];
- The modding out of transformations is very basis dependent;
- It is not invariant under duality;
- They are difficult to compute with.

We will present an alternative construction for a model of the realization space that will be suitable to some different applications.
Section 2

Slack variety of a polytope
Let $P$ be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$. 

The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ given by

$$S_P(i,j) = h_j(p_i).$$
Let $P$ be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ given by

$$S_P(i, j) = h_j(p_i).$$
Let \( P \) be a polytope with facets given by \( h_1(x) \geq 0, \ldots, h_f(x) \geq 0 \), and vertices \( p_1, \ldots, p_v \).

The slack matrix of \( P \) is the matrix \( S_P \in \mathbb{R}^{v \times f} \) given by
\[
S_P(i, j) = h_j(p_i).
\]

Regular hexagon.
Let $P$ be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ given by

$$S_P(i, j) = h_j(p_i).$$

Regular hexagon.

Its $6 \times 6$ slack matrix.

$$
\begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 0 & 1 \\
1 & 2 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0
\end{bmatrix}
$$
Let $P$ be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ given by

$$S_P(i,j) = h_j(p_i).$$

Regular hexagon. Its $6 \times 6$ slack matrix.

$$
\begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 0 & 1 \\
1 & 2 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0
\end{bmatrix}
$$

- The slack matrix is defined only up to column scaling;
Let $P$ be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ given by

$$S_P(i,j) = h_j(p_i).$$

Regular hexagon. Its $6 \times 6$ slack matrix.

\begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 0 & 1 \\
1 & 2 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0
\end{bmatrix}

- The slack matrix is defined only up to column scaling;
- The slack matrix can’t see affine transformations;
- Moreover $P$ is affinely equivalent to the convex hull of the rows of $S_P$. 

If $P$ is a $d$-polytope with $\mathcal{V}$-representation $\{p_1, \ldots, p_v\}$ and $\mathcal{H}$-representation $Ax \leq b$ then

$$S_P = \begin{bmatrix} b & -A \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_v \end{bmatrix}$$

In particular $S_P$ has rank $d + 1$. 
Characterization of slack matrices

If $P$ is a $d$-polytope with $V$-representation $\{p_1, \ldots, p_v\}$ and $H$-representation $Ax \leq b$ then

$$S_P = \begin{bmatrix} b & -A \\ p_1 & 1 \\ p_2 & 1 \\ \vdots & \vdots \\ p_v & 1 \end{bmatrix}$$

In particular $S_P$ has rank $d + 1$.

Any polytope of the same combinatorial class of $P$ must have a slack matrix with the same zero-pattern.
Characterization of slack matrices

If $P$ is a $d$-polytope with $V$-representation $\{p_1, \ldots, p_v\}$ and $H$-representation $Ax \leq b$ then

$$S_P = \begin{bmatrix} b \quad -A \\ p_1 \quad p_2 \quad \cdots \quad p_v \end{bmatrix}$$

In particular $S_P$ has rank $d + 1$.

Any polytope of the same combinatorial class of $P$ must have a slack matrix with the same zero-pattern.

**Theorem (GGKPRT, 2013)**

A nonnegative matrix $S$ is the slack matrix of some realization of $P$ if and only if

1. $\text{supp}(S) = \text{supp}(S_P)$;
2. $\text{rank}(S) = \text{rank}(S_P) = d + 1$;
3. the all ones vector lies in the column span of $S$. 

João Gouveia (UC)
Characterization of slack matrices

If $P$ is a $d$-polytope with $\mathcal{V}$-representation $\{p_1, \ldots, p_v\}$ and $\mathcal{H}$-representation $Ax \leq b$ then

$$S_P = [b - A] \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_v \end{bmatrix}$$

In particular $S_P$ has rank $d + 1$.

Any polytope of the same combinatorial class of $P$ must have a slack matrix with the same zero-pattern.

**Theorem (GGKPRT, 2013)**

A nonnegative matrix $S$ is the slack matrix of some realization of $P$ if and only if

1. $\text{supp}(S) = \text{supp}(S_P)$;
2. $\text{rank}(S) = \text{rank}(S_P) = d + 1$;
3. the all ones vector lies in the column span of $S$.

There is a one-to-one correspondence between matrices with those properties (up to column scaling) and realizations of $P$ (up to affine equivalence).
In general, we will be interested in modding out projective transformations.

\[ Q \overset{p}{=} P \iff Q = \phi(P), \quad \phi(x) = \frac{Ax + b}{c^\top + d}, \quad \det \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix} \neq 0 \]
In general, we will be interested in modding out \textit{projective transformations}.

\[ Q \overset{p}{=} P \iff Q = \phi(P), \quad \phi(x) = \frac{Ax + b}{c^\top + d}, \quad \det \begin{bmatrix} A & b \\ c^\top x & d \end{bmatrix} \neq 0 \]

All convex quadrilaterals are projectively equivalent to a square. A square is \textit{projectively unique}.
Projective equivalence

In general, we will be interested in modding out projective transformations.

\[ Q \stackrel{p}{=} P \iff Q = \phi(P), \quad \phi(x) = \frac{Ax + b}{c^\top + d}, \quad \det \begin{bmatrix} A & b \\ c^\top x & d \end{bmatrix} \neq 0 \]

All convex quadrilaterals are projectively equivalent to a square. A square is projectively unique.

Slack matrices offer a natural way of quotient projective transformations.

Theorem (GPRT, 2017)

\[ Q \stackrel{p}{=} P \iff S_Q = D_v S_P D_f \text{ for some positive diagonal matrices } D_v, D_f \]
Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle.$$
Slack ideals

Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)^\infty.$$
Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)^\infty.$$
Slack ideals

**Slack ideal**

Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)_{\infty}.$$ 

$$S_P = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$$
Slack ideals

Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)^\infty.$$
Slack ideals

Slack ideal

Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)\infty.$$
Slack ideals

**Slack ideal**

Let $P$ be a $d$-polytope and $S_P(x)$ a symbolic matrix with the same support as $S_P$. Then the slack ideal of $P$ is

$$I_P = \langle (d + 2)\text{-minors of } S_P(x) \rangle : (\prod x_i)^\infty.$$

$$S_P = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad S_P(x) = \begin{pmatrix}
x_1 & x_2 & 0 & 0 & 0 \\
0 & x_3 & x_4 & 0 & 0 \\
0 & 0 & x_5 & x_6 & 0 \\
x_7 & 0 & 0 & x_8 & 0 \\
0 & 0 & 0 & 0 & x_9
\end{pmatrix}$$

$$I_P = \langle x_1x_3x_5x_8x_9 - x_2x_4x_6x_7x_9 \rangle : (\prod x_i)^\infty = \langle x_1x_3x_5x_8 - x_2x_4x_6x_7 \rangle$$
\( \mathcal{V}(I_P) \) is the **slack variety** of \( P \).

- Positive part of slack variety: \( \mathcal{V}_+(I_P) = \mathcal{V}(I_P) \cap \mathbb{R}^n_+ \)
Slack realization space

- $\mathcal{V}(I_P)$ is the **slack variety** of $P$.
- Positive part of slack variety: $\mathcal{V}^+(I_P) = \mathcal{V}(I_P) \cap \mathbb{R}^n_+$
- $\mathbb{R}^v_+ \times \mathbb{R}^f_+$ acts on $\mathcal{V}^+(I_P)$:

$$D_v s D_f \in \mathcal{V}^+(I_P) \quad \text{for every } s \in \mathcal{V}^+(I_P),$$

$D_v, D_f$ positive diagonal matrices
\( \mathcal{V}(I_P) \) is the **slack variety** of \( P \).

Positive part of slack variety: \( \mathcal{V}_+(I_P) = \mathcal{V}(I_P) \cap \mathbb{R}^n_+ \)

\( \mathbb{R}^v_+ \times \mathbb{R}^f_+ \) acts on \( \mathcal{V}_+(I_P) \):

\[
D_v s D_f \in \mathcal{V}_+(I_P) \quad \text{for every } s \in \mathcal{V}_+(I_P),
\]

\( D_v, D_f \) positive diagonal matrices

---

**Theorem (GMTW, 2017)**

\[
\mathcal{V}_+(I_P)/(\mathbb{R}^v_+ \times \mathbb{R}^f_+) \overset{1:1}{\leftrightarrow} \text{classes of projectively equivalent polytopes of the same combinatorial type as } P.
\]
\( \mathcal{V}(I_P) \) is the **slack variety** of \( P \).

Positive part of slack variety:  
\[
\mathcal{V}^+_+(I_P) = \mathcal{V}(I_P) \cap \mathbb{R}^n_+
\]

\( \mathbb{R}^v_+ \times \mathbb{R}^f_+ \) acts on \( \mathcal{V}(I_P) \):

\[
D_v s D_f \in \mathcal{V}^+_+(I_P) \quad \text{for every } s \in \mathcal{V}^+_+(I_P), \\
D_v, D_f \text{ positive diagonal matrices}
\]

**Theorem (GMTW, 2017)**

\[
\mathcal{V}^+_+(I_P)/(\mathbb{R}^v_+ \times \mathbb{R}^f_+) \xleftarrow{1:1} \text{classes of projectively equivalent polytopes of the same combinatorial type as } P.
\]

We call \( \mathcal{V}^+_+(I_P)/(\mathbb{R}^v_+ \times \mathbb{R}^f_+) \) the **slack realization space** of \( P \).
Connection to the classical model

\[ x = \begin{bmatrix} p_1 & \cdots & p_v \end{bmatrix} \in \mathcal{R}(P) \]
Connection to the classical model

\[ x = [p_1 \cdots p_v] \in \mathcal{R}(P) \quad \rightarrow \quad \bar{x} = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix} \]
Connection to the classical model

\[ x = \begin{bmatrix} p_1 & \cdots & p_v \end{bmatrix} \in \mathcal{R}(P) \rightarrow \quad \bar{x} = \begin{bmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_v \end{bmatrix} \]

\[\downarrow\]

row space of \( \bar{x} \in \text{Gr}_{d+1}(\mathbb{R}^v) \)
Connection to the classical model

\[ x = [p_1 \cdots p_v] \in \mathcal{R}(P) \quad \rightarrow \quad \bar{x} = \begin{bmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_v \end{bmatrix} \]

\[ \tilde{x} = (\det(\bar{x}_I))_I \in \mathbb{P}^{\binom{v}{d}-1} \quad \leftarrow \quad \text{row space of } \bar{x} \in \text{Gr}_{d+1}(\mathbb{R}^v) \]
Connection to the classical model

\[
x = \begin{bmatrix} p_1 & \cdots & p_v \end{bmatrix} \in \mathcal{R}(P) \quad \rightarrow \quad \bar{x} = \begin{bmatrix} 1 & \cdots & 1 \\
p_1 & \cdots & p_v \end{bmatrix}
\]

\[
\tilde{x} = (\det(\bar{x}_I))_I \in \mathbb{P}^{(v)}_{d-1} \quad \leftarrow \quad \text{row space of } \bar{x} \in \text{Gr}_{d+1}(\mathbb{R}^v)
\]

This sends \( \mathcal{R}(P) \) bijectively up to affine transformations into a subset of the Plücker embedding of \( \text{Gr}_{d+1}(\mathbb{R}^v) \) cut out (mostly) from positivity, negativity and nullity conditions on some of the variables.
Connection to the classical model

\[ x = [p_1 \cdots p_v] \in \mathcal{R}(P) \quad \Rightarrow \quad \bar{x} = \begin{bmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_v \end{bmatrix} \]

\[ \tilde{x} = (\det(\bar{x}_I))_I \in \mathbb{P}^{d-1} \quad \leftarrow \quad \text{row space of } \bar{x} \in \text{Gr}_{d+1}(\mathbb{R}^v) \]

This sends \( \mathcal{R}(P) \) bijectively up to affine transformations into a subset of the Plücker embedding of \( \text{Gr}_{d+1}(\mathbb{R}^v) \) cut out (mostly) from positivity, negativity and nullity conditions on some of the variables.

If for every facet \( k \) of \( P \) we pick a set \( I_k \) of \( d - 1 \) spanning vertices we can define a matrix

\[ (S(\tilde{x}))_{k,l} = \pm \tilde{x}(I_k,l) \]

This is a slack matrix of \( P \) and its row space is \( \bar{x} \).
Section 3

Applications
Application 1: Psd-minimality

A semidefinite representation of size $k$ of a $d$-polytope $P$ is a description

$$P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where $A_i$ and $B_i$ are $k \times k$ real symmetric matrices.
Application 1: Psd-minimality

A semidefinite representation of size $k$ of a $d$-polytope $P$ is a description

$$P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where $A_i$ and $B_i$ are $k \times k$ real symmetric matrices.

If we allow $A_i$ and $B_i$ to be hermitian, we call it a complex semidefinite representation.
Application 1: Psd-minimality

A semidefinite representation of size $k$ of a $d$-polytope $P$ is a description

$$P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where $A_i$ and $B_i$ are $k \times k$ real symmetric matrices.

If we allow $A_i$ and $B_i$ to be hermitian, we call it a complex semidefinite representation.

Projection on $x_1$ and $x_2$ of

$$\begin{bmatrix}
1 & x_1 & x_2 \\
x_1 & x_1 & y \\
x_2 & y & x_2
\end{bmatrix} \succeq 0.$$
Application 1: Psd-minimality

A semidefinite representation of size $k$ of a $d$-polytope $P$ is a description

$$P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where $A_i$ and $B_i$ are $k \times k$ real symmetric matrices.

If we allow $A_i$ and $B_i$ to be hermitian, we call it a complex semidefinite representation.

Projection on $x_1$ and $x_2$ of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$
A semidefinite representation of size \( k \) of a \( d \)-polytope \( P \) is a description

\[ P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\} \]

where \( A_i \) and \( B_i \) are \( k \times k \) real symmetric matrices.

If we allow \( A_i \) and \( B_i \) to be hermitian, we call it a complex semidefinite representation.

Projection on \( x_1 \) and \( x_2 \) of

\[
\begin{bmatrix}
1 & x_1 & x_2 \\
x_1 & x_1 & y \\
x_2 & y & x_2
\end{bmatrix} \succeq 0.
\]

Optimizing over such sets is “easy”: we want small representations.
Application 1: Psd-minimality

A semidefinite representation of size $k$ of a $d$-polytope $P$ is a description

$$P = \left\{ x \in \mathbb{R}^d \mid \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where $A_i$ and $B_i$ are $k \times k$ real symmetric matrices.

If we allow $A_i$ and $B_i$ to be hermitian, we call it a complex semidefinite representation.

Projection on $x_1$ and $x_2$ of

$$\begin{bmatrix}
1 & x_1 & x_2 \\
x_1 & x_1 & y \\
x_2 & y & x_2
\end{bmatrix} \succeq 0.$$

Optimizing over such sets is “easy”: we want small representations. Turns out the smallest possible size is $d + 1$. When does that happen?
Application 1: Psd-minimality (part 2)

Theorem (GRT 2013; GGS 2016)

- A polytope $P$ is psd-minimal $\iff \exists S_p(y) \in \mathcal{V}_\mathbb{R}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_\mathbb{C}$-minimal $\iff \exists S_p(y) \in \mathcal{V}_\mathbb{C}(I_P)$ such that $S_P = S_P(|y|^2)$.

Lemma

- If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.
  - In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
  - In $\mathbb{R}^4$ (31 types) this allowed the classification [GPRT, 2017].

Lemma

- Suppose $P$ is psd$_\mathbb{C}$-minimal, i.e. $S_P = S_P(|y|^2)$.
  - If $I_P$ has a trinomial $x^a + x^b - x^c$ then $\mathbb{R}(y^ay^b) = 0$.
    - In $\mathbb{R}^2$ (3 types), [GGS 2017, CG 2018].
    - In $\mathbb{R}^3$ who knows?...
**Application 1: Psd-minimality (part 2)**

**Theorem (GRT 2013; GGS 2016)**

- A polytope $P$ is psd-minimal $\iff \exists S_p(y) \in V_{\mathbb{R}}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_\mathbb{C}$-minimal $\iff \exists S_p(y) \in V_{\mathbb{C}}(I_P)$ such that $S_P = S_P(|y|^2)$.

**Lemma** If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.
Application 1: Psd-minimality (part 2)

**Theorem (GRT 2013; GGS 2016)**

- A polytope $P$ is psd-minimal $\iff \exists S_P(y) \in V_{\mathbb{R}}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_{\mathbb{C}}$-minimal $\iff \exists S_P(y) \in V_{\mathbb{C}}(I_P)$ such that $S_P = S_P(|y|^2)$

**Lemma** If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.

- In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
Application 1: Psd-minimality (part 2)

Theorem (GRT 2013; GGS 2016)

- A polytope $P$ is psd-minimal $\iff \exists S_p(y) \in \mathcal{V}_\mathbb{R}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_\mathbb{C}$-minimal $\iff \exists S_p(y) \in \mathcal{V}_\mathbb{C}(I_P)$ such that $S_P = S_P(|y|^2)$

Lemma If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.

- In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
- In $\mathbb{R}^4$ (31 types) this allowed the classification [GPRT, 2017].
Application 1: Psd-minimality (part 2)

**Theorem (GRT 2013; GGS 2016)**

- A polytope $P$ is psd-minimal $\Leftrightarrow \exists S_p(y) \in V_\mathbb{R}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_\mathbb{C}$-minimal $\Leftrightarrow \exists S_p(y) \in V_\mathbb{C}(I_P)$ such that $S_P = S_P(|y|^2)$

**Lemma** If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.

- In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
- In $\mathbb{R}^4$ (31 types) this allowed the classification [GPRT, 2017].

**Lemma** Suppose $P$ is psd$_\mathbb{C}$-minimal, i.e. $S_P = S_P(|y|^2)$. If $I_P$ has a trinomial $x^a + x^b - x^c$ then $\Re(y^a y^b) = 0$. 
Application 1: Psd-minimality (part 2)

Theorem (GRT 2013; GGS 2016)

- A polytope $P$ is psd-minimal $\iff \exists S_P(y) \in \mathcal{V}_{\mathbb{R}}(I_P)$ such that $S_P = S_P(y^2)$.
- A polytope $P$ is psd$_\mathbb{C}$-minimal $\iff \exists S_P(y) \in \mathcal{V}_{\mathbb{C}}(I_P)$ such that $S_P = S_P(|y|^2)$.

Lemma  If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.

- In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
- In $\mathbb{R}^4$ (31 types) this allowed the classification [GPRT, 2017].

Lemma  Suppose $P$ is psd$_\mathbb{C}$-minimal, i.e. $S_P = S_P(|y|^2)$.

If $I_P$ has a trinomial $x^a + x^b - x^c$ then $\Re(y^a y^b) = 0$.

- In $\mathbb{R}^2$ (3 types), [GGS 2017, CG 2018].
Application 1: Psd-minimality (part 2)

Theorem (GRT 2013; GGS 2016)

A polytope $P$ is psd-minimal $\iff \exists S_p(y) \in \mathcal{V}_R(I_P)$ such that $S_P = S_P(y^2)$.

A polytope $P$ is psd$_C$-minimal $\iff \exists S_p(y) \in \mathcal{V}_C(I_P)$ such that $S_P = S_P(|y|^2)$.

Lemma If $I_P$ has a trinomial $x^a + x^b - x^c$ then $P$ is not psd-minimal.

- In $\mathbb{R}^2$ (2 types), $\mathbb{R}^3$ (6 types) this recovers [GRT 2013].
- In $\mathbb{R}^4$ (31 types) this allowed the classification [GPRT, 2017].

Lemma Suppose $P$ is psd$_C$-minimal, i.e. $S_P = S_P(|y|^2)$. If $I_P$ has a trinomial $x^a + x^b - x^c$ then $\Re(y^a \overline{y}^b) = 0$.

- In $\mathbb{R}^2$ (3 types), [GGS 2017, CG 2018].
- In $\mathbb{R}^3$ who knows?...
Application 2: Rationality

A combinatorial polytope is *rational* if it has a realization in which all vertices have rational coordinates.

Scaling rows and columns to set some variables to 1 (this does not affect rationality):

$x_{46}^2 + x_{46} - 1 \in \mathcal{I}_P \Rightarrow x_{46} = -1 \pm \sqrt{5}$

There are no rational realizations.

This can be extended to the ideal of the Perles polytope (d=8, v=12, f=34).

It is not rational but also its slack ideal is not prime.
Application 2: Rationality

A combinatorial polytope is *rational* if it has a realization in which all vertices have rational coordinates.

**Lemma**  
A polytope $P$ is rational $\iff \mathcal{V}_+(I_P)$ has a rational point.
Application 2: Rationality

A combinatorial polytope is *rational* if it has a realization in which all vertices have rational coordinates.

**Lemma** A polytope $P$ is rational $\Leftrightarrow \mathcal{V}_+(I_P)$ has a rational point.

We consider the following point-line arrangement in the plane [Grünbaum, 1967]:

\[
S_P(x) = \begin{pmatrix}
  x_1 & 0 & x_2 & 0 & x_3 & x_4 & x_5 & x_6 & 0 \\
  x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\
  x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\
  x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\
  x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\
  0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\
  0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\
  0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\
  0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0 
\end{pmatrix}
\]

Scaling rows and columns to set some variables to 1 (this does not affect rationality):

\[
x_{46} \in I_P \Rightarrow x_{46} = -\frac{1}{\pm \sqrt{5}} \Rightarrow \text{no rational realizations}
\]

This can be extended to the ideal of the Perles polytope ($d=8$, $v=12$, $f=34$)

It is not rational but also its slack ideal is not prime.
Application 2: Rationality

A combinatorial polytope is **rational** if it has a realization in which all vertices have rational coordinates.

**Lemma**  
A polytope $P$ is rational $\iff V_+(I_P)$ has a rational point.

We consider the following point-line arrangement in the plane [Grüenbaum, 1967]:

$$S_P(x) = \begin{pmatrix} x_1 & 0 & x_2 & 0 & x_3 & 0 & x_4 & x_5 & x_6 & 0 \\ x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\ x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\ x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\ x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\ 0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\ 0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\ 0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\ 0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0 \end{pmatrix}$$

Scaling rows and columns to set some variables to 1 (this does not affect rationality):

$$x_{46}^2 + x_{46} - 1 \in I_P$$
Application 2: Rationality

A combinatorial polytope is *rational* if it has a realization in which all vertices have rational coordinates.

**Lemma** A polytope $P$ is rational $\iff \mathcal{V}_+(I_P)$ has a rational point.

We consider the following point-line arrangement in the plane [Grüumbaum, 1967]:

$$
S_P(x) = \begin{pmatrix}
    x_1 & 0 & x_2 & 0 & x_3 & x_4 & x_5 & x_6 & 0 \\
    x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\
    x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\
    x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\
    x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\
    0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\
    0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\
    0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\
    0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0
\end{pmatrix}
$$

Scaling rows and columns to set some variables to 1 (this does not affect rationality):

$$
2x_{46}^2 + x_{46} - 1 \in I_P \Rightarrow x_{46} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \text{no rational realizations}
$$
Application 2: Rationality

A combinatorial polytope is **rational** if it has a realization in which all vertices have rational coordinates.

**Lemma** A polytope $P$ is rational $\iff \mathcal{V}_+(I_P)$ has a rational point.

We consider the following point-line arrangement in the plane [Grünbaum, 1967]:

$$S_P(x) = \begin{pmatrix}
    x_1 & 0 & x_2 & 0 & x_3 & x_4 & x_5 & x_6 & 0 \\
    x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\
    x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\
    x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\
    x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\
    0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\
    0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\
    0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\
    0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0
\end{pmatrix}$$

Scaling rows and columns to set some variables to 1 (this does not affect rationality):

$$x_{46}^2 + x_{46} - 1 \in I_P \Rightarrow x_{46} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \text{no rational realizations}$$

This can be extended to the ideal of the Perles polytope ($d=8, v=12, f=34$)
It is not rational but also its slack ideal is not prime.
Application 3: Realizability

**Steinitz problem** Check whether an abstract polytopal complex is the boundary of an actual polytope.
Application 3: Realizability

Steinitz problem  Check whether an abstract polytopal complex is the boundary of an actual polytope.

[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

\[
S_P(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\
0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\
0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\
x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} & 0 \\
x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\
x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\
x_{30} & x_{31} & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 & 0
\end{pmatrix}.
\]
### Steinitz problem

*Check whether an abstract polytopal complex is the boundary of an actual polytope.*

[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

\[
S_P(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\
0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\
0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\
x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} & 0 \\
x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\
x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\
x_{30} & x_{31} & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 & 0 \\
\end{pmatrix}
\]

### Proposition

*P is realizable*  \iff  \( V_+(I_P) \neq \emptyset \).
Application 3: Realizability

**Steinitz problem** Check whether an abstract polytopal complex is the boundary of an actual polytope.

[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

\[
S_P(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\
0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & x_{13} \\
0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\
x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} \\
x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\
x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\
x_{30} & x_{31} & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 & 0 \\
\end{pmatrix}.
\]

**Proposition** \( P \) is realizable \( \iff V_+(I_P) \neq \emptyset \).

In this case, \( I_P = \langle 1 \rangle \Rightarrow \) no rank 5 matrix with this support \( \Rightarrow \) no polytope.
Section 4

One more application
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?
How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?

For $n = 2$, clearly $\dim(\mathcal{R}(P)) = 2v$. 
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?

For $n = 2$, clearly $\dim(\mathcal{R}(P)) = 2v$.

For $n = 3$ we have $\dim(\mathcal{R}(P)) = v + f + 4$. [Steinitz]
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?

For $n = 2$, clearly $\dim(\mathcal{R}(P)) = 2v$.

For $n = 3$ we have $\dim(\mathcal{R}(P)) = v + f + 4$. [Steinitz]

For $n > 3$ there are very few general results/tools.
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?

For $n = 2$, clearly $\dim(\mathcal{R}(P)) = 2v$.

For $n = 3$ we have $\dim(\mathcal{R}(P)) = v + f + 4$. [Steinitz]

For $n > 3$ there are very few general results/tools.

$$\dim(\mathcal{R}(P)) \leftrightarrow \dim(\mathcal{V}_+(I_P))$$
Dimension of the realization space

How much freedom does a certain combinatorial structure give us?

Given a polytope $P \subseteq \mathbb{R}^n$, what is the dimension of $\mathcal{R}(P)$?

For $n = 2$, clearly $\dim(\mathcal{R}(P)) = 2v$.

For $n = 3$ we have $\dim(\mathcal{R}(P)) = v + f + 4$. [Steinitz]

For $n > 3$ there are very few general results/tools.

$$\dim(\mathcal{R}(P)) \leftrightarrow \dim(\mathcal{V}(I_P))$$

Can we compute the dimension of $\mathcal{V}(I_P)$?
How to do this?

1. Exact Computational Algebra

   Too hard: $V(I_P)$ has around $v \times f$ entries.

2. Statistical topology from samples

   Implies a sufficiently representative sample of polytopes with a given combinatorial structure.

   Hopeless in general.

3. Maybe we can use the structure of the variety to do enough?

João Gouveia (UC)
Slack variety of a polytope and its applications
ICERM 2018
How to do this?

Exact Computational Algebra

1. Exact Computational Algebra

2. Statistical topology from samples

3. Implies a sufficiently representative sample of polytopes with a given combinatorial structure.

However, hopeless in general.

Maybe we can use the structure of the variety to do enough?
How to do this?

**Exact Computational Algebra**

Too hard: $\mathcal{V}(I_P)$ has around $v \times f$ entries.
How to do this?

1. **Exact Computational Algebra**
   Too hard: $\mathcal{V}(I_P)$ has around $v \times f$ entries.

2. **Statistical topology from samples**
How to do this?

1. **Exact Computational Algebra**
   Too hard: $\mathcal{V}(I_P)$ has around $v \times f$ entries.

2. **Statistical topology from samples**
   Implies a sufficiently representative sample of polytopes with a given combinatorial structure.
How to do this?

1. **Exact Computational Algebra**
   Too hard: $\mathcal{V}(I_P)$ has around $v \times f$ entries.

2. **Statistical topology from samples**
   Implies a sufficiently representative sample of polytopes with a given combinatorial structure. Hopeless in general.
How to do this?

1. **Exact Computational Algebra**
   Too hard: $\mathcal{V}(I_P)$ has around $v \times f$ entries.

2. **Statistical topology from samples**
   Implies a sufficiently representative sample of polytopes with a given combinatorial structure. Hopeless in general.

However

3. Maybe we can use the structure of the variety to do enough?
Perturbing a polytope

Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?
Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?

Given a polytope $P$, we can always add noise to the entries of $S_P$ but then we are away from $\mathcal{V}(I_P)$. 
Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?

Given a polytope $P$, we can always add noise to the entries of $S_P$ but then we are away from $\mathcal{V}(I_P)$. Can we project it back?
Perturbing a polytope

Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?

Given a polytope $P$, we can always add noise to the entries of $S_P$ but then we are away from $\mathcal{V}(I_P)$. Can we project it back? Yes!!! By using the fact that

$$\mathcal{V}(I_P) = \{X : \text{rank}(X) \leq d + 1\} \cap L.$$
Perturbing a polytope

Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?

Given a polytope $P$, we can always add noise to the entries of $S_P$ but then we are away from $\mathcal{V}(I_P)$. Can we project it back? Yes!!! By using the fact that

$$\mathcal{V}(I_P) = \{X : \text{rank}(X) \leq d + 1\} \cap L.$$ 

Proto-theorem - GPP sometime in the future

In general, Dykstra’s alternate projection algorithm will applied to $\tilde{S} = S_P + \text{noise}$ will converge to the projection of $\tilde{S}$ in $\mathcal{V}(I_P)$. 
Perturbing a polytope

Let us go to a related more basic problem:

How to perturb a polytope while preserving the combinatorics?

Given a polytope $P$, we can always add noise to the entries of $S_P$ but then we are away from $\mathcal{V}(I_P)$. Can we project it back? Yes!!! By using the fact that

$$\mathcal{V}(I_P) = \{X : \text{rank}(X) \leq d + 1\} \cap L.$$ 

Proto-theorem - GPP sometime in the future

In general, Dykstra’s alternate projection algorithm will applied to $\bar{S} = S_P + \text{noise}$ will converge to the projection of $\bar{S}$ in $\mathcal{V}(I_P)$.

This is not a full answer to the question, but might be enough.
Idea:

1. Start with $\mathbf{S} \in \mathbb{V}(\mathbb{I}P)$;
2. Add noise to each entry following $\mathcal{N}(0, \epsilon)$ distribution;
3. Project the perturbed point to $\mathbf{x}$ in the variety and record the distance to $\mathbf{S}$;
4. Repeat ad nauseam

What is happening?

As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $\mathbf{S}$.

Proto-theorem - GPP sometime in the future

As $\epsilon \to 0$, $\frac{1}{\epsilon^2}d(\mathbf{x}, \mathbf{S})^2 \to \chi^2(\text{dim} \mathbb{V}(\mathbb{I}P))$.

In particular the average distance squared should converge to the dimension!
Idea:

1. Start with $S_P \in \mathcal{V}_\mathbb{R}(I_P)$;

As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$.

Proto-theorem - GPP sometime in the future

As $\epsilon \to 0$, $\frac{1}{2} \epsilon^2 d(x, S_P)^2 \to \chi_2(\dim \mathcal{V}_\mathbb{R}(I_P))$.

In particular the average distance squared should converge to the dimension!
Idea:

1. Start with $S_P \in \mathcal{V}_\mathbb{R}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;

What is happening? As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$.

Proto-theorem - GPP

As $\epsilon \to 0$, $\frac{1}{\epsilon^2} d(x, S_P)^2 \to \chi_2(\dim \mathcal{V}_\mathbb{R}(I_P))$.

In particular the average distance squared should converge to the dimension!
Idea:

1. Start with $S_P \in \mathcal{V}_{\mathbb{R}}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;
3. Project the perturbed point to $x$ in the variety and record the distance to $S_P$;

What is happening?

As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$.

Proto-theorem - GPP

As $\epsilon \to 0$,

$$\frac{\epsilon^2}{2} d(x, S_P)^2 \to \chi^2(\dim \mathcal{V}_{\mathbb{R}}(I_P)).$$

In particular the average distance squared should converge to the dimension!
Idea:

1. Start with $S_P \in \mathcal{V}_{\mathbb{R}}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;
3. Project the perturbed point to $x$ in the variety and record the distance to $S_P$;
4. Repeat *ad nauseam*
Enter the statistics

Idea:
1. Start with $S_P \in \mathcal{V}_{\mathbb{R}}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;
3. Project the perturbed point to $x$ in the variety and record the distance to $S_P$;
4. Repeat ad nauseam

What is happening?
As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$. 

Proto-theorem - GPP

\[ \text{Proto-theorem - GPP} \]

As $\epsilon \to 0$, 
\[ \epsilon^2 d(x, S_P) \to \chi_2(\text{dim} \mathcal{V}_{\mathbb{R}}(I_P)). \]

In particular the average distance squared should converge to the dimension!
Idea:
1. Start with $S_P \in \mathcal{V}_\mathbb{R}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;
3. Project the perturbed point to $x$ in the variety and record the distance to $S_P$;
4. Repeat *ad nauseam*

What is happening?
As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$.

Proto-theorem - GPP *sometime in the future*

As $\epsilon \to 0$,

$$\frac{1}{\epsilon^2} d(x, S_P)^2 \to \chi^2(\dim \mathcal{V}_\mathbb{R}(I_P)).$$
Idea:
1. Start with $S_P \in \mathcal{V}_{\mathbb{R}}(I_P)$;
2. Add noise to each entry following $N(0, \epsilon)$ distribution;
3. Project the perturbed point to $x$ in the variety and record the distance to $S_P$;
4. Repeat ad nauseam

What is happening?
As $\epsilon \to 0$ we are essentially projecting onto the tangent space in $S_P$.

Proto-theorem - GPP sometime in the future
As $\epsilon \to 0$,
\[
\frac{1}{\epsilon^2} d(x, S_P)^2 \to \chi^2(\dim \mathcal{V}_{\mathbb{R}}(I_P)).
\]

In particular the average distance squared should converge to the dimension!
Recall that the hypersimplex $H_{n,k}$ is defined as $\{x \in [0,1]^n : \sum x_i = k\}$.

Theorem (Padrol-Sanyal 2016)
Let $I_{n,k}$ be the slack ideal of $H_{n,k}$. For $k \geq 2$, we have
\[
\dim V + (I_{n,k}) \leq (n-1) + (nk) + 2n-1
\]
with equality for $k = 2$. 

Let's try it out.
Recall that the hypersimplex $H_{n,k}$ is defined as

$$H_{n,k} = \{ x \in [0, 1]^n : \sum x_i = k \}.$$
Recall that the hypersimplex $H_{n,k}$ is defined as

$$H_{n,k} = \{ x \in [0, 1]^n : \sum x_i = k \}.$$

**Theorem (Padrol-Sanyal 2016)**

Let $I_{n,k}$ be the slack ideal of $H_{n,k}$. For $k \geq 2$, we have

$$\dim V_+(I_{n,k}) \leq \binom{n-1}{2} + \binom{n}{k} + 2n - 1$$

with equality for $k = 2$. 

---

João Gouveia (UC)  
Slack variety of a polytope and its applications  
ICERM 2018 25 / 27
Recall that the hypersimplex $H_{n,k}$ is defined as

$$H_{n,k} = \{ x \in [0, 1]^n : \sum x_i = k \}. $$

**Theorem (Padrol-Sanyal 2016)**

Let $I_{n,k}$ be the slack ideal of $H_{n,k}$. For $k \geq 2$, we have

$$\dim V_+(I_{n,k}) \leq \binom{n-1}{2} + \binom{n}{k} + 2n - 1$$

with equality for $k = 2$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16/16.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>25/25.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>36/36.0</td>
<td>41/41.0</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>49/49.0</td>
<td>63/63.0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>64/64.1</td>
<td>92/91.8</td>
<td>106/105.9</td>
</tr>
<tr>
<td>9</td>
<td>81/81.0</td>
<td>129/129.0</td>
<td>171/171.0</td>
</tr>
</tbody>
</table>
Given a poset $P$ with base elements $\{1, \ldots, n\}$ its order polytope is

$$\{x \in \mathbb{R}^n : 0 \leq x_i \leq x_j \leq 1 \forall i \leq_P j\}.$$
Given a poset $P$ with base elements $\{1, \ldots, n\}$ its order polytope is

$$\{x \in \mathbb{R}^n : 0 \leq x_i \leq x_j \leq 1 \forall i \leq_P j\}.$$  

**Conjecture (Bogart, Chaves)**

The order polytope is projectively unique if and only if there is no antichain bigger than two.
Given a poset $P$ with base elements $\{1, \ldots, n\}$ its order polytope is

$$\{x \in \mathbb{R}^n : 0 \leq x_i \leq x_j \leq 1 \forall i \leq_P j\}.$$  

**Conjecture (Bogart, Chaves)**

The order polytope is projectively unique if and only if there is no antichain bigger than two.

We checked a few dozen examples and we saw $\dim(R(P)) = 0$ up to one decimal case everytime there was no large antichain.
Given a poset $P$ with base elements $\{1, \ldots, n\}$ its order polytope is

$$\{ x \in \mathbb{R}^n : 0 \leq x_i \leq x_j \leq 1 \forall i \leq_P j \}.$$  

**Conjecture (Bogart, Chaves)**

The order polytope is projectively unique if and only if there is no antichain bigger than two.

We checked a few dozen examples and we saw $\dim(R(P)) = 0$ up to one decimal case everytime there was no large antichain.

We tried many three dimensional polytopes, projectively unique polytopes and pretty much everything we could get our hands on. All worked.
Conclusion

There are many more questions, and a more algebraic perspective.
There are many more questions, and a more algebraic perspective.

For further reading:

- arXiv:1708.04739 - *The Slack Realization Space of a Polytope*
- arXiv:1808.01692 - *Projectively unique polytopes and toric slack ideal*

with Antonio Macchia, Rekha Thomas and Amy Wiebe.
Conclusion

There are many more questions, and a more algebraic perspective.

For further reading:

- arXiv:1708.04739 - The Slack Realization Space of a Polytope
- arXiv:1808.01692 - Projectively unique polytopes and toric slack ideal

with Antonio Macchia, Rekha Thomas and Amy Wiebe.

Thank you